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GUANGSHI LÜ, JIE WU & WENGUANG ZHAI

Abstract. In this paper we study several divisor problems related to the Epstein zeta-function. We are able to improve previous results and establish some new results by applying some classical techniques.

1. Introduction

In this paper, we shall continue our study on divisor problems related to the Epstein zeta-function [12, 13, 14]. Let $\ell \geqslant 2$, $\mathbf{y} := (y_1, \dots, y_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \,(\mathrm{mod}\, 2)$ for $1 \leqslant i \leqslant \ell$. Thus a positive definite quadratic form $Q(\mathbf{y})$ can be written as

$$Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y} = \frac{1}{2}\sum_{1\leqslant i\leqslant \ell} a_{ii}y_i^2 + \sum_{1\leqslant i<j\leqslant \ell} a_{ij}y_iy_j,$$

where $\mathbf{y}^{\mathrm{t}}$ is the transpose of $\mathbf{y}$. The corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$Z_Q(s) := \sum_{\mathbf{y}\in\mathbb{Z}^\ell\smallsetminus\{\mathbf{0}\}} \frac{1}{Q(\mathbf{y})^s} = \sum_{n\geqslant 1} \frac{r(n,Q)}{n^s} \tag{1.1}$$

for $\Re e\, s > \ell/2$, where

$$r(n,Q) := \big|\big\{\mathbf{y}\in\mathbb{Z}^\ell : Q(\mathbf{y}) = n\big\}\big|.$$

According to [21], $Z_Q(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ with only a simple pole at $s = \ell/2$, and satisfies a functional equation of Riemann type.

For each integer $k \geqslant 1$, we are interested in the mean value of the k-fold Dirichlet convolution of $r(n,Q)$ defined by

$$r_k(n,Q) := \sum_{n_1\cdots n_k = n} r(n_1,Q)\cdots r(n_k,Q). \tag{1.2}$$

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The study on the asymptotic behavior of the error term

$$\Delta_k^*(x,Q) := \sum_{n\leqslant x} r_k(n,Q) - \operatorname*{Res}_{s=\ell/2}\big(Z_Q(s)^k x^s s^{-1}\big) \tag{1.3}$$

has received much attention [11, 3, 21]. In particular Sankaranarayanan [21] showed, by the complex method, that for $k \geqslant 2$ and $\ell \geqslant 3$,

$$\Delta_k^*(x,Q) \ll x^{\ell/2-1/k+\varepsilon}, \tag{1.4}$$

where and throughout this paper ε denotes an arbitrarily small positive constant.

Recently inspired by Iwaniec's book [8, Chapter 11], Lü [12] noted that (1.4) can be improved for the quadratic forms of level one. These quadratic forms verify the following supplementary conditions:

$$\ell \equiv 0 \,(\mathrm{mod}\, 8), \qquad \mathbf{A} \text{ is equivalent to } \mathbf{A}^{-1}, \qquad |\mathbf{A}| = 1.$$

For such quadratic forms, we have [8, (11.32)]

$$r(n,Q) = \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}\sigma_{\ell/2-1}(n) + a_f(n,Q) \quad (n \geqslant 1), \tag{1.5}$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, $\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function and $a_f(n,Q)$ is the n-th Fourier coefficient of a cusp form $f(z,Q)$ of weight $\ell/2$ with respect to the full modular group $\mathrm{SL}(2,\mathbb{Z})$ verifying Deligne's bound

$$|a_f(n,Q)| \leqslant n^{(\ell/2-1)/2}\sigma_0(n) \qquad (n \geqslant 1). \tag{1.6}$$

Thus

$$Z_Q(s) = \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}\zeta(s)\zeta(s-\ell/2+1) + L(s,f) \tag{1.7}$$

for $s \in \mathbb{C}\smallsetminus\{\ell/2\}$, where $L(s,f)$ is the Hecke L-function associated with $f(z,Q)$. In view of basic properties of $\zeta(s)$ and $L(s,f)$, it is not difficult to see that $\zeta(s-\ell/2+1)$ is more dominant and we may view $\Delta_k^*(Q;x)$ as the classical k-dimensional divisor problem associated to the Riemann zeta-function. With the help of these ideas, Lü, Wu & Zhai [13] obtained, via a simple convolution argument,

$$\Delta_k^*(x,Q) \ll x^{\ell/2-1+\theta_k+\varepsilon} \quad (x \geqslant 2) \tag{1.8}$$

for $k = 2, 3$, † where θ_k is the exponent in the classical k-dimension divisor problem

$$\sum_{n\leqslant x} \tau_k(n) = \operatorname*{Res}_{s=1}(\zeta(s)^k x^s s^{-1}) + O\big(x^{\theta_k+\varepsilon}\big) \qquad (x \geqslant 2). \tag{1.9}$$

Besides, an Ω-result for $k = 2, 3$ and a mean value theorem for $\Delta_2^*(x,Q)$ have been established in [13] and [14], respectively.

In this paper we shall refine Sankaranarayanan's (1.4) for general positive definite quadratic forms Q. In this case, it is known that [8, Theorem 11.2]

$$r(n,Q) = \frac{(2\pi)^{\ell/2}}{\Gamma(\ell/2)\sqrt{|\mathbf{A}|}} n^{\ell/2-1}\sigma(n,Q) + O\big(n^{\ell/4-\delta_\ell+\varepsilon}\big) \tag{1.10}$$

†When $k \geqslant 4$, a similar result has been proved by Lü [12] with the complex method.

for $\ell \geqslant 4$, where $\mathrm{e}(t) := \mathrm{e}^{2\pi \mathrm{i} t}$ $(t \in \mathbb{R})$,

$$S(Q) := \sum_{0 \leqslant y_1, \dots, y_\ell \leqslant q-1} \mathrm{e}(Q(\mathbf{y})),$$

$$\sigma(n, Q) := \sum_{q=1}^{\infty} \frac{1}{q^\ell} \sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big) \mathrm{e}\Big(-\frac{hn}{q}\Big),$$

and

$$\delta_\ell := \begin{cases} \frac{1}{4} & \text{if } \ell \text{ is odd,} \\ \frac{1}{2} & \text{if } \ell \text{ is even.} \end{cases}$$

Here and in the sequel, the symbol $\sum^*$ means $\sum_{(h,q)=1}$. Here we propose two methods to bound $\Delta_k^*(x, Q)$: the complex method and the convolution method. The former allows us to establish nontrivial estimates for $\Delta_k^*(x, Q)$ for all $k \geqslant 1$ and $\ell \geqslant 4$. But the convolution argument is more powerful for $k = 1, 2, 3$ when $\ell \geqslant 6$.

Let

$$L_Q(s) := \sum_{n=1}^{\infty} \frac{\sigma(n, Q)}{n^s} \quad (\Re e\, s > 1). \tag{1.11}$$

In view of the bound (cf [8, Lemma 10.5])

$$S(hQ/q) \ll q^{\ell/2} \quad ((h, q) = 1), \tag{1.12}$$

The Dirichlet series $L_Q(s)$ is absolutely convergent for $\Re e\, s > 1$ provided $\ell \geqslant 5$. In Section 2 we shall prove that $L_Q(s)$ can be analytically continued to a meromorphic function on the half plane $\Re e\, s > 0$, which has a simple pole at $s = 1$ with residue 1 (see Lemma 2.1 below), and establish some individual and average subconvexity bounds for $L_Q(s)$ similar to $\zeta(s)$ (see Lemmas 2.2 and 2.3 below). With the help of these new tools, the standard complex method allows us to deduce the following result, which improves Sankaranarayanan's (1.4) when $k \geqslant 3$.

Theorem 1. *Let $\ell \geqslant 4$ and $k \geqslant 1$. We have*

$$\Delta_k^*(x, Q) \ll x^{\ell/2 - 1 + \vartheta_{k,\ell} + \varepsilon} \quad (x \geqslant 2), \tag{1.13}$$

where

$$\vartheta_{k,\ell} = \begin{cases} 1/2 & \text{if } 1 \leqslant k \leqslant 4 \text{ and } \ell \geqslant 4, \\ k/(k+4) & \text{if } 5 \leqslant k \leqslant 12 \text{ and } \ell = 4 \text{ or } \ell \geqslant 6, \\ (13k-4)/(13k+44) & \text{if } 5 \leqslant k \leqslant 12 \text{ and } \ell = 5, \\ (k-3)/k & \text{if } 13 \leqslant k \leqslant 49 \text{ and } \ell = 4 \text{ or } \ell \geqslant 6, \\ (4k-11)/(4k+1) & \text{if } 13 \leqslant k \leqslant 49 \text{ and } \ell = 5, \\ 1 - (2738k^2)^{-1/3} & \text{if } k \geqslant 50 \text{ and } \ell \geqslant 4. \end{cases}$$

The convolution argument of [13] can be also generalized to estimate $\Delta_k^*(x, Q)$. Though (1.10) is more complicated than (1.5), we can use it to establish a connection between $\Delta_k^*(x, Q)$ and the divisor problem with congruence conditions. We will discuss this divisor problem in Section 4. For $\mathbf{q} := (q_1, \dots, q_k) \in \mathbb{N}^k$ and $\mathbf{r} :=$

$(r_1, \dots, r_r) \in \mathbb{N}^k$ such that $r_i \leqslant q_i$ $(1 \leqslant i \leqslant k)$, define

$$\tau_k(n; \mathbf{q}, \mathbf{r}) := \sum_{\substack{n_1 \cdots n_k = n \\ n_i \equiv r_i (\operatorname{mod} q_i)\, (1 \leqslant i \leqslant k)}} 1, \qquad D_k(x; \mathbf{q}, \mathbf{r}) := \sum_{n \leqslant x} \tau_k(n; \mathbf{q}, \mathbf{r}).$$

The divisor problem with congruence conditions aims to bound the error term

$$\Delta_k(x; \mathbf{q}, \mathbf{r}) := D_k(x; \mathbf{q}, \mathbf{r}) - \operatorname*{Res}_{s=1} \big(\zeta(s, r_1/q_1) \cdots \zeta(s, r_k/q_k) x^s s^{-1}\big) \tag{1.14}$$

where $\zeta(s, \alpha)$ is the Hurwitz zeta-function defined by

$$\zeta(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} \quad (0 < \alpha \leqslant 1, \sigma > 1). \tag{1.15}$$

With the help of the convolution argument, we can prove the following result, which offers better exponents than (1.4) for $k = 1, 2, 3$ when $\ell \geqslant 6$.

Theorem 2. *Let $\ell \geqslant 6$ and $k = 1, 2, 3$. Assume that there is some $\vartheta_k \in (0, 1)$ such that*

$$\Delta_k(x; \mathbf{q}, \mathbf{r}) \ll_{k,\ell,\varepsilon} \big(x/(q_1 \cdots q_k)\big)^{\vartheta_k + \varepsilon}$$

uniformly for $1 \leqslant r_i \leqslant q_i$ $(1 \leqslant i \leqslant k)$ and $q_1 \cdots q_k \leqslant x$. Then we have

$$\Delta_k^*(x, Q) \ll_{k,\ell,\varepsilon} x^{\ell/2 - 1 + \vartheta_k + \varepsilon}.$$

Especially we can take

$$\vartheta_k = \begin{cases} 0 & \textit{if } k = 1, \\ \frac{131}{416} & \textit{if } k = 2, \\ \frac{43}{96} & \textit{if } k = 3. \end{cases} \tag{1.16}$$

Another interesting problem related to $r(n, Q)$ is to evaluate its kth power sum. In this direction, Landau [11] first showed that

$$\sum_{n \leqslant x} r(n, Q) = \frac{(2\pi)^{\ell/2}}{\Gamma(\ell/2 + 1)\sqrt{|\det Q|}} x^{\ell/2} + O\big(x^{\ell/2 - \ell/(\ell+1)}\big). \tag{1.17}$$

For $k = 2$, Müller [16] proved that

$$\sum_{n \leqslant x} r(n, Q)^2 = \begin{cases} A_Q x \log x + B_Q x + O\big(x^{3/5} \log x\big) & \text{if } \ell = 2, \\ C_Q x^{\ell - 1} + O\big(x^{\ell - 1 - 2(\ell-1)/(4\ell-3)}\big) & \text{if } \ell \geqslant 3, \end{cases} \tag{1.18}$$

where A_Q, B_Q and C_Q are some constants depending on Q. In this paper we study a more general correlated sum of $r(n, Q)$, which contains the kth power sum as a special case.

Theorem 3. *Let $\ell \geqslant 5$, $k \geqslant 1$ and $a_1, \dots, a_k$ be fixed non-negative integers. Then*

$$\sum_{n \leqslant x} \prod_{1 \leqslant i \leqslant k} r(n + a_i, Q) = C_Q(a_1, \dots, a_k) x^{(\ell/2-1)k+1} + O_{a_1,\dots,a_k}\big(x^{(\ell/2-1)k + \eta_\ell(\varepsilon)}\big),$$

where $C_Q(a_1, \dots, a_k)$ is a constant depending on Q and $(a_1, \dots, a_k)$, and

$$\eta_\ell(\varepsilon) := \begin{cases} \frac{1}{2} + \varepsilon & \text{if } \ell = 5, \\ \varepsilon & \text{if } \ell = 6, 7, \\ 0 & \text{if } \ell \geqslant 8. \end{cases}$$

Obviously the two particular cases of Theorem 3:

$$\text{“}k = 1,\ a_1 = 0\text{”} \qquad \text{and} \qquad \text{“}k = 2,\ a_1 = a_2 = 0\text{”}$$

improve (1.17) for $\ell \geqslant 6$ and (1.18) for $\ell \geqslant 5$, respectively. It is worth to indicate that our method is different from Müller [16] and simpler.

As an application of Theorem 3, we give the following asymptotic formula for the correlated sum involving the divisor sum function $\sigma_{\ell/2-1}(n)$.

Corollary 1. *Let $8 \mid \ell$, $k \geqslant 2$ and $a_1, \dots, a_k$ be fixed non-negative integers. Then*

$$\sum_{n \leqslant x} \prod_{1 \leqslant i \leqslant k} \sigma_{\ell/2-1}(n + a_i) = D_\ell(a_1, \dots, a_k) x^{(\ell/2-1)k+1} + O_{a_1,\dots,a_k}\big(x^{(\ell/2-1)k}\big),$$

where $D_\ell(a_1, \dots, a_k)$ is a constant depending on ℓ and $a_1, \dots, a_k$.

2. Study on $L_Q(s)$

This section will be devoted to study $L_Q(s)$, which is important in the proof of Theorem 1.

Lemma 2.1. *If $\ell \geqslant 5$, then $L_Q(s)$ can be analytically continued to a meromorphic function on the half plane $\Re e\, s > 0$, which has a simple pole at $s = 1$ with residue 1.*

Proof. By using the definition of $\sigma(n, Q)$, a simple calculation shows that

$$\begin{aligned} L_Q(s) &= \sum_{q=1}^{\infty} \frac{1}{q^\ell} \sideset{}{^*}\sum_{h=1}^{q} S(hQ/q) F(s, -h/q) \\ &= \zeta(s) + \sum_{q=2}^{\infty} \frac{1}{q^\ell} \sideset{}{^*}\sum_{h=1}^{q} S(hQ/q) F(s, -h/q) \end{aligned} \tag{2.1}$$

for $\Re e\, s > 1$, where $F(s, a)$ is the periodic zeta-function defined by

$$F(s, a) := \sum_{n=1}^{\infty} \frac{\mathrm{e}(an)}{n^s} \qquad (\Re e\, s > 1).$$

In view of well-known proprieties of $\zeta(s)$, it suffices to prove that the last double series in (2.1) can be continued analytically to the half plane $\Re e\, s > 0$.

Introducing the notation

$$M(u, \alpha) := \sum_{n \leqslant u} \mathrm{e}(n\alpha) \ll \min\big\{u, \|\alpha\|^{-1}\big\}, \tag{2.2}$$

where $\|\alpha\| := \min_{t\in\mathbb{Z}} |\alpha - t|$, a simple integration by parts allows us to write, for $\Re e\, s > 1$, $q \geqslant 2$, and $(h, q) = 1$, that

$$F(s, h/q) = \sum_{n\leqslant |t|+1} \frac{\mathrm{e}(hn/q)}{n^s} - \frac{M(|t|+1, h/q)}{(|t|+1)^s} + s\int_{|t|+1}^{\infty} \frac{M(u, h/q)}{u^{s+1}}\,\mathrm{d}u.$$

This formula and (2.2) give an analytic continuation of $F(s, h/q)$ to the region $\Re e\, s > 0$ and the estimate

$$F(s, h/q) \ll \frac{|t|+1}{\|h/q\|}$$

holds uniformly for $\Re e\, s > 0$. From this and (1.12), we deduce that

$$\begin{aligned}\sum_{q=2}^{\infty} \frac{1}{q^\ell} \sideset{}{^*}\sum_{h=1}^{q} |S(hQ/q)F(s, -h/q)| &\ll \sum_{q=2}^{\infty} \frac{|t|+1}{q^{\ell/2}} \sum_{h=1}^{q/2} \frac{q}{h}\\ &\ll (|t|+1) \sum_{q=2}^{\infty} \frac{\log q}{q^{\ell/2-1}},\end{aligned}$$

which absolutely converges for $\Re e\, s > 0$ since $\ell \geqslant 5$. □

The next two lemmas give individual and average subconvexity bounds for $L_Q(s)$.

Lemma 2.2. *Let $\ell \geqslant 5$ and $\varepsilon > 0$. We have*

$$L_Q(\sigma + \mathrm{i}t) \ll \min\left\{|t|^{(1-\sigma)/3+\varepsilon},\ |t|^{18.4974(1-\sigma)^{3/2}}(\log |t|)^{2/3}\right\} \tag{2.3}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1, |t| \geqslant 2$.

Proof. According to [20, pp 127], we have, for $0 < \alpha \leqslant 1$, that

$$\begin{aligned}F(s, \alpha) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big\{&\mathrm{e}^{\frac{\pi \mathrm{i}}{2}(1-s)}\zeta^*(1-s, \alpha) + \mathrm{e}^{\frac{\pi \mathrm{i}}{2}(1-s)}\alpha^{-(1-s)}\\ &+ \mathrm{e}^{-\frac{\pi \mathrm{i}}{2}(1-s)}\zeta^*(1-s, 1-\alpha) + \mathrm{e}^{\frac{\pi \mathrm{i}}{2}(1-s)}(1-\alpha)^{-(1-s)}\Big\},\end{aligned} \tag{2.4}$$

where $\zeta^*(s, \alpha) := \zeta(s, \alpha) - \alpha^{-s}$ and $\zeta(s, \alpha)$ is the Hurwitz zeta-function defined by (1.15). By combining (2.4) with Stirling's formula, we have, for $s = \frac{1}{2} + \mathrm{i}t$ and $(h, q) = 1$ with $q \geqslant 2$, that

$$\begin{aligned}F(s, h/q) \ll\ &\zeta^*(\tfrac{1}{2} - \mathrm{i}t, h/q) + \zeta^*(\tfrac{1}{2} - \mathrm{i}t, 1 - h/q)\\ &+ q^{1/2}h^{-1/2} + q^{1/2}(q-h)^{-1/2}.\end{aligned} \tag{2.5}$$

Similar to the Riemann zeta-function, it is known that [2, Theorem]

$$\zeta^*(s, \alpha) \ll (|t|+1)^{(1-\sigma)/3+\varepsilon} \tag{2.6}$$

and

$$\zeta^*(s, \alpha) \ll |t|^{18.4974(1-\sigma)^{3/2}}(\log |t|)^{2/3} \tag{2.7}$$

uniformly for $0 < \alpha \leqslant 1$, $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $|t| \geqslant 10$(see e.g. [22] and [10], respectively). Now the required estimate (2.3) follows from (2.1), (2.4), (2.5), (2.6) and (2.7), by noticing that

$$\sum_{q\geqslant 2} \frac{1}{q^\ell} \sideset{}{^*}\sum_{h=1}^{q} |S(hQ/q)|\big(q^{1/2}h^{-1/2} + q^{1/2}(q-h)^{-1/2}\big) \ll \sum_{q\geqslant 2} \frac{1}{q^{\ell/2-1}} \ll 1, \tag{2.8}$$

since $\ell \geqslant 5$. □

Lemma 2.3. *Let $\ell \geqslant 5$ and $k \geqslant 1$ be fixed integers. Then we have*

$$\int_1^T |L_Q(\tfrac{1}{2}+\mathrm{i}t)|^k \,\mathrm{d}t \ll T^{1+\beta_{k,\ell}+\varepsilon}, \tag{2.9}$$

where

$$\beta_{k,\ell} := \begin{cases} 0 & \text{if } 1 \leqslant k \leqslant 4 \text{ and } \ell \geqslant 5, \\ 13(k-4)/96 & \text{if } 5 \leqslant k \leqslant 12 \text{ and } \ell = 5, \\ (k-4)/8 & \text{if } 5 \leqslant k \leqslant 12 \text{ and } \ell \geqslant 6, \\ k/6 - 11/12 & \text{if } k > 12 \text{ and } \ell = 5, \\ k/6 - 1 & \text{if } k > 12 \text{ and } \ell \geqslant 6. \end{cases}$$

Proof. Write $s = \frac{1}{2} + it$. In order to prove Lemma 2.3, it suffices to prove that

$$\int_1^T |L_Q(s)|^4 \,\mathrm{d}t \ll T^{1+\varepsilon}, \tag{2.10}$$

$$\int_1^T |L_Q(s)|^{12} \,\mathrm{d}t \ll T^{2+\max\{(16-3\ell)/12,\,0\}+\varepsilon}. \tag{2.11}$$

Our key tools are the fourth mean value of Hurwitz' zeta-function[1, Theorem 4]

$$\int_1^T |\zeta^*(s,\alpha)|^4 \,\mathrm{d}t \ll T(\log T)^{10}, \tag{2.12}$$

which holds uniformly for $0 < \alpha \leqslant 1$, $T \geqslant 2$, and the twelfth power moment of the Dirichlet L-function(see [15])

$$\sum_{\chi\,(\bmod\, q)} \int_1^T |L(s,\chi)|^{12} \,\mathrm{d}t \ll q^3 T^{2+\varepsilon}, \tag{2.13}$$

which holds uniformly for $q \geqslant 1$, $T \geqslant 2$.

From (2.1), (2.5) and (2.8), we deduce that

$$|L_Q(s)| \ll |\zeta(s)| + \sum_{q\geqslant 2} \frac{1}{q^{\ell/2}} \sum_{h\leqslant q/2} |\zeta^*(\tfrac{1}{2}-\mathrm{i}t, h/q)| + 1. \tag{2.14}$$

So by Hölder's inequality we have

$$|L_Q(s)|^4 \ll |\zeta(s)|^4 + \bigg(\sum_{q\geqslant 2}\sum_{h\leqslant q/2} \frac{1}{q^{5/2}}\bigg)^3 \sum_{q\geqslant 2}\sum_{h\leqslant q/2} \frac{|\zeta^*(\frac{1}{2}-\mathrm{i}t, h/q)|^4}{q^{(4\ell-15)/2}} + 1, \tag{2.15}$$

which combining (2.12) leads to (2.10) since $\ell \geqslant 5$.

In order to prove (2.11), we write, by the orthogonality relation of Dirichlet characters,

$$\begin{aligned} F(s,h/q) &= \sum_{a=1}^{q} \mathrm{e}(ah/q) \sum_{n\equiv a (\bmod\, q)} \frac{1}{n^s} \\ &= \frac{1}{\varphi(q)} \sum_{\chi\,(\bmod\, q)} G(h,\overline{\chi}) L(s,\chi), \end{aligned}$$

where $\varphi(q)$ is the Euler function and $G(h,\chi)$ is the Gauss sum defined by

$$G(h,\chi) := \sum_{a=1}^{q} \chi(a)\mathrm{e}(ah/q).$$

By virtue of the well-known bound $|G(h,\chi)| \leqslant q^{1/2}$ $((h,q)=1)$, it follows that

$$F(s,h/q) \ll \frac{q^{1/2}}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} |L(s,\chi)| \tag{2.16}$$

Let $\eta > 0$ be a parameter to be chosen later. We split the sum over q in (2.1) into two parts according to $q \leqslant T^\eta$ or $q > T^\eta$. Using (2.16) for $q \leqslant T^\eta$ and (2.5), (2.8) for $q > T^\eta$, we deduce that

$$|L_Q(s)| \ll L_{Q,1}(s) + L_{Q,2}(s) + 1 \tag{2.17}$$

where

$$L_{Q,1}(s) := \sum_{q \leqslant T^\eta} \frac{1}{q^{(\ell-1)/2}} \sum_{\chi \,(\mathrm{mod}\, q)} |L(s,\chi)|,$$
$$L_{Q,2}(s) := \sum_{q > T^\eta} \frac{1}{q^{\ell/2}} \sum_{h \leqslant q/2} |\zeta^*(\tfrac{1}{2} - \mathrm{i}t, h/q)|.$$

By Hölder's inequality again we have

$$\begin{aligned} |L_{Q,1}(s)|^{12} &\ll \bigg(\sum_{q \leqslant T^\eta} \sum_{\chi (\mathrm{mod}\, q)} \frac{1}{q^2} \bigg)^{11} \sum_{q \leqslant T^\eta} \sum_{\chi (\mathrm{mod}\, q)} \frac{|L(s,\chi)|^{12}}{q^{6\ell-28}} \\ &\ll (\log T)^{11} \sum_{q \leqslant T^\eta} \sum_{\chi (\mathrm{mod}\, q)} \frac{|L(s,\chi)|^{12}}{q^{6\ell-28}}, \end{aligned}$$

which combining (2.12) gives

$$\int_1^T |L_{Q,1}(s)|^{12} \,\mathrm{d}t \ll T^{2+\varepsilon} \sum_{q \leqslant T^\eta} q^{-6\ell+31} \ll T^{2+\max\{\eta(32-6\ell),\, 0\}+\varepsilon}. \tag{2.18}$$

The bound (2.6) implies trivially that

$$\sum_{q > T^\eta} \frac{1}{q^{\ell/2}} \sum_{h \leqslant q/2} |\zeta^*(\tfrac{1}{2} - \mathrm{i}t, h/q)| \ll T^{1/6+\varepsilon} \sum_{q > T^\eta} \frac{1}{q^{\ell/2-1}} \ll T^{1/6-\eta(\ell/2-2)+\varepsilon}.$$

On the other hand, similarly to (2.15), we have

$$\bigg(\sum_{q > T^\eta} \frac{1}{q^{\ell/2}} \sum_{h \leqslant q/2} |\zeta^*(\tfrac{1}{2} - \mathrm{i}t, h/q)| \bigg)^4 \ll T^{-3\eta/2} \sum_{q > T^\eta} \sum_{h \leqslant q/2} \frac{|\zeta^*(\frac{1}{2} - \mathrm{i}t, h/q)|^4}{q^{(4\ell-15)/2}}.$$

Combining these with (2.12) yields that

$$\int_1^T |L_{Q,2}(s)|^{12} \mathrm{d}t \ll T^{8\{1/6-\eta(\ell/2-2)\}-3\eta/2-\eta(4\ell-19)/2+1+\varepsilon} \ll T^{7/3-(6\ell-24)\eta+\varepsilon}. \tag{2.19}$$

Now (2.11) follows from (2.18) and (2.19) with the choice of $\eta = \frac{1}{24}$. $\square$

3. Proof of Theorem 1

3.1. The case $\ell \geqslant 5$ and $1 \leqslant k \leqslant 49$.

By [21, Lemmas 3.1 and 3.2], it follows that

$$\sum_{n\leqslant x} r_k(n,Q) = \frac{1}{2\pi \mathrm{i}}\int_{\ell/2+\varepsilon-\mathrm{i}T}^{\ell/2+\varepsilon+\mathrm{i}T} Z_Q(s)^k \frac{x^s}{s}\,\mathrm{d}s + O\biggl(\frac{x^{\ell/2+\varepsilon}}{T} + x^{\varepsilon}\biggr). \tag{3.1}$$

In view of (1.10) and Lemma 2.1, we have

$$Z_Q(s) \ll |L_Q(s-\ell/2+1)| + 1 \tag{3.2}$$

uniformly for $\Re e\, s \geqslant (\ell+3)/4+\varepsilon$ and $t \neq 0$. By noticing that $(\ell+3)/4 \leqslant (\ell-1)/2$ (since $\ell \geqslant 5$), we can move the integration in (3.1) to the parallel segment with $\Re e\, s = (\ell-1)/2+\varepsilon$. By Lemma 2.1 and the residue theorem, we have

$$\frac{1}{2\pi \mathrm{i}}\int_{\ell/2+\varepsilon-\mathrm{i}T}^{\ell/2+\varepsilon+\mathrm{i}T} Z_Q(s)^k \frac{x^s}{s}\,\mathrm{d}s = \operatorname*{Res}_{s=\ell/2}\bigl(Z_Q(s)^k x^s s^{-1}\bigr) - \int_{\mathscr{L}} Z_Q(s)^k \frac{x^s}{s}\,\mathrm{d}s, \tag{3.3}$$

where $\mathscr{L}$ is the contour joining $\ell/2+\mathrm{i}T$, $(\ell-1)/2+\varepsilon+\mathrm{i}T$, $(\ell-1)/2+\varepsilon-\mathrm{i}T$, $\ell/2-\mathrm{i}T$ with straight lines. With the help of (3.2) and Lemmas 2.2-2.3, the contribution of the horizontal segments to the last integral of (3.3) is

$$\ll x^{\ell/2+\varepsilon}T^{-1} \tag{3.4}$$

provided $T \leqslant x^{3/k}$ $(2 \leqslant k \leqslant 49)$, and the contribution of the vertical segment is

$$\begin{aligned} &\ll x^{(\ell-1)/2+\varepsilon}\int_1^T \frac{\bigl|L_Q(\frac12+\varepsilon+\mathrm{i}t)\bigr|^k}{t}\,\mathrm{d}t \\ &\ll x^{(\ell-1)/2+\varepsilon}T^{\beta_{k,\ell}+\varepsilon}. \end{aligned} \tag{3.5}$$

Combining (3.4), (3.5) and (3.3) with (3.1) and taking $T = x^{1/(2+2\beta_{k,\ell})}$, we obtain the required estimate for $\ell \geqslant 5$ and $k \leqslant 49$.

3.2. The case $\ell \geqslant 5$ and $k \geqslant 50$.

In this case we apply Lemma 2.2. After applying Perron's formula, we move the integration to the parallel segment with $\Re e\, s = \sigma_0 = \ell/2 - 2Ak^{-2/3}$ and choose $T = x^{Ak^{-2/3}}$, where $A > 0$ is an absolute constant which will be determined later. By applying (3.2) and Lemma 2.2, the contribution of the vertical segment is

$$\begin{aligned} &\ll x^{\ell/2-2Ak^{-2/3}}T^{18.5k\{\ell/2-(\sigma_0-\ell/2+1)\}^{3/2}}(\log x)^{2k/3+1} \\ &= x^{\ell/2-(2A-18.5\times\sqrt{8}A^{5/2})k^{-2/3}}(\log x)^{2k/3+1}, \end{aligned}$$

and the contribution of the horizontal segments is

$$\begin{aligned} &\ll x^{\ell/2+\varepsilon}T^{-1}(\log x)^{2k/3} + \max_{\sigma_0\leqslant\sigma\leqslant\ell/2} x^{\sigma}T^{18.5k\{1-(\sigma-\ell/2+1)\}^{3/2}-1}(\log x)^{2k/3} \\ &\ll \bigl(x^{\ell/2-(A-\varepsilon)k^{-2/3}} + x^{\ell/2-(2A-37\sqrt{2}A^{5/2})k^{-2/3}}\bigr)(\log x)^{2k/3+1}. \end{aligned}$$

Now we choose A to satisfy $A = 2A - 37\sqrt{2}A^{5/2}$, which gives $A = 2738^{-1/3}$. Therefore for $k \geqslant 50$ we have

$$\Delta_k^*(x,Q) \ll x^{\ell/2-(2738k^2)^{-1/3}}(\log x)^{2k/3+1}.$$

3.3. The case $\ell = 4$.

It is known that in this case

$$\theta(z, Q) := \sum_{n=0}^{\infty} r(n, Q)\mathrm{e}(nz)$$

is a modular form of weight 2 and level N (N is an integer such that $N\mathbf{A}^{-1}$ is also an integral matrix, see [8, Theorem 10.9]). Then by the standard theory of modular forms, $Z_Q(s)$ can be written as

$$Z_Q(s) = L_Q(s) + L(s, f),$$

where $L_Q(s)$ is a linear combination of series of the form

$$(t_1 t_2)^{-s} L(s, \chi_1) L(s - \ell/2 + 1, \chi_2),$$

and $L(s, f)$ is the Hecke L-function associated with a cusp form of weight 2 and level N. Here t_1, t_2 are positive divisors of N, and χ_1, χ_2 are Dirichlet characters modulo N/t_1, N/t_2 respectively.

According to (1.6) with $\ell = 4$, we learn that $|L(s, f)| \ll_\varepsilon 1$ for $\Re e\, s \geqslant \frac{3}{2} + \varepsilon$. When $\ell = 4$, we also have $\frac{\ell}{2} - \frac{1}{2} = \frac{3}{2}$. Therefore similar to (3.2), we have

$$|Z_Q(s)| \ll |L_Q(s)| + 1$$

for $\Re e\, s \geqslant \frac{3}{2} + \varepsilon$. On recalling the classical results ‡

$$L(\tfrac{1}{2} + \mathrm{i}t, \chi) \ll (|t| + 1)^{1/6+\varepsilon}, \tag{3.6}$$

$$L(\tfrac{1}{2} + \mathrm{i}t, \chi) \ll (|t| + 1)^{18.4974(1-\sigma)^{3/2}} (\log |t|)^{2/3}, \tag{3.7}$$

$$\int_1^T |L(\tfrac{1}{2} + \mathrm{i}t, \chi)|^4 \,\mathrm{d}t \ll T^{1+\varepsilon}, \tag{3.8}$$

$$\int_1^T |L(\tfrac{1}{2} + \mathrm{i}t, \chi)|^{12} \,\mathrm{d}t \ll T^{2+\varepsilon}, \tag{3.9}$$

it is easy to see that the estimates in Lemmas 2.2 and 2.3 also hold when $\ell = 4$. Thus we can follow the arguments of Section 3.1 to show that (1.13) also is true for $\ell = 4$. We omit the details.

4. The divisor problem with congruence conditions

The divisor problem with congruence conditions (1.14) was first studied by Nowak [18, 19] and Menzer & Nowak [17]. They established very interesting Ω-type results for $\Delta_k(x; \mathbf{q}, \mathbf{r})$. As they indicated ([18, page 456; page 110], [17, Remarks]), it is straightforward to obtain the same O-results as in the classical divisor problem, since the theory of $\zeta(s)$ developed in the textbooks [22, 7] may be readily generalized to L-series. Here we state this O-result as a lemma, since it is important in the proof of Theorem 2.

‡(3.6) is a special case of [5, Corollary 1]; (3.7) can be deduced easily from (2.7); (3.9) is a consequence of (2.13).

Lemma 4.1. *Suppose $k = 1, 2, 3$. We have*

$$D_k(x; \mathbf{q}, \mathbf{r}) = \frac{x}{q_1 \cdots q_k} \mathcal{P}_{k-1}\left(\log \frac{x}{q_1 \cdots q_k}\right) + O_{k,\varepsilon}\left(\left(\frac{x}{q_1 \cdots q_k}\right)^{\vartheta_k + \varepsilon}\right)$$

uniformly for $x \geqslant 3$, $1 \leqslant r_i \leqslant q_i$ $(1 \leqslant i \leqslant k)$ and $q_1 \cdots q_k \leqslant x$, where $\mathcal{P}_{k-1}(t)$ is a polynomial of degree $k-1$ and ϑ_k is given by (1.16). Furthermore we have

$$\max|\text{coefficients of } \mathcal{P}_{k-1}| \ll \sum_{1 \leqslant i_1 < \cdots < i_{k-1} \leqslant k} \frac{q_{i_1} \cdots q_{i_{k-1}}}{r_{i_1} \cdots r_{i_{k-1}}}. \tag{4.1}$$

Proof. It is easy to see that

$$\begin{aligned} D_k(x; \mathbf{q}, \mathbf{r}) &= \sum_{\substack{1 \leqslant n_1 \cdots n_k \leqslant x \\ n_i \equiv r_i (\operatorname{mod} q_i)\, (1 \leqslant i \leqslant k)}} 1 \\ &= \sum_{\substack{m_1 \geqslant 0, \ldots, m_k \geqslant 0 \\ (m_1 + r_1/q_1) \cdots (m_k + r_k/q_k) \leqslant x/(q_1 \cdots q_k)}} 1. \end{aligned}$$

Thus the case of $k = 1$ is trivial. When $k = 2$, we can write from the above formula by the well-known hyperbolic approach, that

$$D_2(x; \mathbf{q}, \mathbf{r}) = (x/q_1 q_2) \mathcal{P}_1(\log(x/q_1 q_2)) + \Delta_2(x; \mathbf{q}, \mathbf{r}),$$

where $\psi(t) := \{t\} - \frac{1}{2}$ ($\{t\}$ is the fractional part of t) and

$$\Delta_2(x; \mathbf{q}, \mathbf{r}) = -\sum_{1 \leqslant i \leqslant 2} \sum_{m_i \leqslant \sqrt{x/(q_1 q_2)} - r_i/q_i} \psi\left(\frac{x/(q_1 q_2)}{m_i + r_i/q_i}\right) + O(1).$$

Using Huxley's new result on exponential sums [6] we get

$$\Delta_2(x; \mathbf{q}, \mathbf{r}) \ll (x/q_1 q_2)^{131/416 + \varepsilon}.$$

For $k = 3$, we could also follow Kolesnik's argument [9] to show $\vartheta_3 = 43/96$.

Next we prove (4.1). When s is near to 1, it is well known that (we suppose $0 < \lambda \leqslant 1$)

$$\zeta(s, \lambda) = \frac{1}{s-1} - \frac{\Gamma'}{\Gamma}(\lambda) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \gamma_n(\lambda)(s-1)^n$$

where $\gamma_n(\lambda)$ is the n-th Stieltjes constant. By the Cauchy formula, it is not difficult to see that $\gamma_n(\lambda) \ll_n 1$ uniformly for $0 < \lambda \leqslant 1$. On the other hand, since $s = 0$ is a pole of order 1 of $\Gamma(s)$, we have

$$\frac{\Gamma'}{\Gamma}(\lambda) \ll \frac{1}{\lambda}.$$

Finally we note that the polynomial $\mathcal{P}_{k-1}$ is determined by

$$\operatorname*{Res}_{s=1}\left(\zeta(s, \lambda_1) \cdots \zeta(s, \lambda_k) x^s s^{-1}\right) = \frac{x}{q_1 \cdots q_k} \mathcal{P}_{k-1}\left(\log \frac{x}{q_1 \cdots q_k}\right).$$

From the above information, we can easily deduce (4.1). $\square$

5. Proof of Theorem 2

In this section for any function $g(n)$ we define

$$g_j(n) := \sum_{n=n_1\cdots n_j} g(n_1)\cdots g(n_j),$$

which is similar to (1.2). Let

$$A := (2\pi)^{\ell/2}/(\Gamma(\ell/2)\sqrt{|\mathbf{A}|}), \quad \tilde{r}(n,Q) := A^{-1}n^{1-\ell/2}r(n,Q).$$

Since $r_k(n,Q) = A^k\tilde{r}_k(n,Q)n^{\ell/2-1}$, it is sufficient to prove that

$$\sum_{n\leqslant x}\tilde{r}_k(n,Q) = x\tilde{P}_{k-1}(\log x) + O_{k,\varepsilon}\big(x^{\vartheta_k+\varepsilon}\big), \tag{5.1}$$

where $\tilde{P}_{k-1}(t)$ is a polynomial of degree $k-1$ and ϑ_k is defined by (1.16).

We first establish the following lemma.

Lemma 5.1. *Suppose $\ell \geqslant 6$ and $k = 1, 2, 3$. Then for any $\varepsilon > 0$, we have*

$$\sum_{n\leqslant x}\sigma_k(n,Q) = xP^*_{k-1}(\log x) + O_{k,\varepsilon}\big(x^{\vartheta_k+\varepsilon}\big), \tag{5.2}$$

where $P^*_{k-1}(t)$ is a polynomial of degree $k-1$ and ϑ_k is defined by (1.16).

Proof. Write

$$\sigma(n,Q) = \tilde{\sigma}(n,Q) + \hat{\sigma}(n,Q),$$

with

$$\tilde{\sigma}(n,Q) := \sum_{q\leqslant x}\frac{1}{q^\ell}\sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big)\mathrm{e}\Big(-\frac{hn}{q}\Big),$$

$$\hat{\sigma}(n,Q) := \sum_{q> x}\frac{1}{q^\ell}\sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big)\mathrm{e}\Big(-\frac{hn}{q}\Big).$$

It is easy to see that $\tilde{\sigma}(n,Q) \ll 1$ and $\hat{\sigma}(n,Q) \ll x^{-1}$ (since $\ell \geqslant 6$). From these facts, we can deduce that

$$\tilde{\sigma}_j(n,Q) \ll \tau_j(n), \qquad \hat{\sigma}_j(n,Q) \ll x^{-j}\tau_j(n)$$

and

$$\begin{aligned}\sigma_k(n,Q) &= \sum_{j=0}^{k}\binom{k}{j}\sum_{dm=n}\tilde{\sigma}_{k-j}(d,Q)\hat{\sigma}_j(m,Q)\\ &= \tilde{\sigma}_k(n,Q) + O(x^{-1}\tau_{k-1}(n)).\end{aligned} \tag{5.3}$$

Thus in order to prove (5.2), it is sufficient to show that

$$\sum_{n\leqslant x}\tilde{\sigma}_k(n,Q) = xP^*_{k-1}(\log x) + O\big(x^{\vartheta_k+\varepsilon}\big). \tag{5.4}$$

By using Lemma 4.1, it follows that

$$\begin{aligned}\sum_{n\leqslant x}\tilde\sigma_k(n,Q) &= \prod_{i=1}^k\sum_{q_i\leqslant x}\frac{1}{q_i^\ell}\sideset{}{^*}\sum_{h_i=1}^{q_i} S\Big(\frac{h_iQ}{q_i}\Big)\sum_{r_i=1}^{q_i}\mathrm{e}\Big(-\frac{h_ir_i}{q_i}\Big)D_k(x;\mathbf{q},\mathbf{r}) \\ &= xS_1(x)+S_2(x)+S_3(x),\end{aligned}\tag{5.5}$$

where

$$S_1(x) := \prod_{i=1}^k\sum_{\substack{q_i\leqslant x\\ q_1\cdots q_k\leqslant x}}\frac{1}{q_i^{\ell+1}}\sideset{}{^*}\sum_{h_i=1}^{q_i} S\Big(\frac{h_iQ}{q_i}\Big)\sum_{r_i=1}^{q_i}\mathrm{e}\Big(-\frac{h_ir_i}{q_i}\Big)\mathcal{P}_{j-1}\Big(\log\frac{x}{q_1\cdots q_k}\Big),$$

$$S_2(x) := \prod_{i=1}^k\sum_{\substack{q_i\leqslant x\\ q_1\cdots q_k> x}}\frac{1}{q_i^{\ell}}\sideset{}{^*}\sum_{h_i=1}^{q_i} S\Big(\frac{h_iQ}{q_i}\Big)\sum_{r_i=1}^{q_i}\mathrm{e}\Big(-\frac{h_ir_i}{q_i}\Big)D_k(x;\mathbf{q},\mathbf{r}),$$

$$S_3(x) := \prod_{i=1}^k\sum_{q_i\leqslant x}\frac{1}{q_i^{\ell}}\sideset{}{^*}\sum_{h_i=1}^{q_i} S\Big(\frac{h_iQ}{q_i}\Big)\sum_{r_i=1}^{q_i}\mathrm{e}\Big(-\frac{h_ir_i}{q_i}\Big)\Delta_k(x;\mathbf{q},\mathbf{r}).$$

It is easy to estimate $S_3(x)$ that

$$S_3(x)\ll x^{\vartheta_k+\varepsilon}\prod_{i=1}^k\sum_{q_i\leqslant x}\frac{1}{q_i^{\ell/2-2+\vartheta_k+\varepsilon}}\ll x^{\vartheta_k+\varepsilon}\qquad(\text{since }\ell\geqslant 6).\tag{5.6}$$

When $q_1\cdots q_k>x$, we use the trivial bound

$$D_k(x;\mathbf{q},\mathbf{r})\ll\sum_{1\leqslant j\leqslant k}\frac{x}{r_1\cdots r_{j-1}q_jr_{j+1}\cdots r_k}+1$$

to write

$$\begin{aligned}S_2(x)&\ll\prod_{i=1}^k\sum_{\substack{q_i\leqslant x\\ q_1\cdots q_k> x}}\frac{1}{q_i^{\ell/2}}\sideset{}{^*}\sum_{h_i=1}^{q_i}\sum_{r_i=1}^{q_i}\bigg\{\sum_{1\leqslant j\leqslant k}\frac{x}{r_1\cdots r_{j-1}q_jr_{j+1}\cdots r_k}+1\bigg\}\\ &\ll x\prod_{i=1}^k\sum_{\substack{q_i\leqslant x\\ q_1\cdots q_k> x}}\frac{\log q_i}{q_i^{\ell/2-1}}+\prod_{i=1}^k\sum_{\substack{q_i\leqslant x\\ q_1\cdots q_k> x}}\frac{1}{q_i^{\ell/2-2}}\\ &\ll x\sum_{n>x}\frac{\tau_k(n)(\log n)^k}{n^{\ell/2-1}}+\sum_{n>x}\frac{\tau_k(n)}{n^{\ell/2-2}}\\ &\ll x^\varepsilon\quad(\text{since }\ell\geqslant 6).\end{aligned}\tag{5.7}$$

Obviously we can write

$$S_1(x)=xP_{k-1}^*(\log x)+O(R(x))\tag{5.8}$$

where

$$R(x):=\prod_{i=1}^k\sum_{\substack{q_i\geqslant 1\\ q_1\cdots q_k> x}}\frac{1}{q_i^{\ell/2-1}}\Big|\mathcal{P}_{k-1}\Big(\log\frac{x}{q_1\cdots q_k}\Big)\Big|.$$

By virtue of (4.1), we deduce that

$$
\begin{aligned}
R(x) &\ll \prod_{i=1}^{k} \sum_{\substack{q_i \geqslant 1 \\ q_1\cdots q_k > x}} \frac{1}{q_i^{\ell/2}} \sum_{r_i=1}^{q_i} \sum_{1\leqslant i_1<\cdots<i_{k-1}\leqslant k} \frac{q_{i_1}\cdots q_{i_{k-1}}}{r_{i_1}\cdots r_{i_{k-1}}} \log^{k-1}(q_1\cdots q_k) \\
&\ll \prod_{i=1}^{k} \sum_{\substack{q_i \leqslant x \\ q_1\cdots q_k > x}} \frac{1}{q_i^{\ell/2-1}} \log^{2j-2}(q_1\cdots q_k) \\
&\ll \sum_{n>x} \frac{\tau_k(n)(\log n)^{2k-2}}{n^{\ell/2-1}} \\
&\ll x^{-\ell/2+2+\varepsilon}.
\end{aligned}
\tag{5.9}
$$

Inserting (5.6), (5.7), (5.8) and (5.9) into (5.5), we obtain (5.4). □

Now we are ready to prove (5.1). By (1.10), we have

$$\tilde r(n,Q) = \sigma(n,Q) + \beta(n) \quad \text{with} \quad \beta(n) = O(n^{-1}).$$

Similar to (5.3), we have

$$\tilde r_k(n,Q) = \sum_{j=0}^{k} \binom{k}{j} \sum_{dm=n} \sigma_j(d,Q)\beta_{k-j}(m), \qquad \beta_j(n) \ll \tau_j(n)/n.$$

Thus Lemma 5.1 allows us to deduce

$$
\begin{aligned}
\sum_{n\leqslant x} \tilde r_k(n,Q) &= \sum_{j=0}^{k} \binom{k}{j} \sum_{m\leqslant x} \beta_{k-j}(m) \sum_{d\leqslant x/m} \sigma_j(d,Q) \\
&= x \sum_{j=0}^{k} \binom{k}{j} \sum_{m\leqslant x} \frac{\beta_{k-j}(m)}{m} P_{j-1}^{*}\Big(\log\frac{x}{m}\Big) + O\big(x^{\vartheta_j+\varepsilon}\big),
\end{aligned}
$$

which implies (5.1) since

$$
\begin{aligned}
\sum_{m\leqslant x} \frac{\beta_{k-j}(m)}{m} P_{j-1}^{*}\Big(\log\frac{x}{m}\Big) &= \sum_{m\geqslant 1} \frac{\beta_{k-j}(m)}{m} P_{j-1}^{*}\Big(\log\frac{x}{m}\Big) + O\big(x^{-1+\varepsilon}\big) \\
&= P_{j-1}^{**}(\log x) + O\big(x^{-1+\varepsilon}\big),
\end{aligned}
$$

where $P_{j-1}^{**}(t)$ is a polynomial of degree $j-1$.

6. Proof of Theorem 3

We reason by recurrence on k. The case of $k=1$ follows from Theorem 1 since a_1 is fixed. Assume that the required asymptotic formula holds for $1, 2, \dots, k-1$. Then in view of (1.10) and the fact that $\ell/4 - \delta_\ell \leqslant \ell/2 - 1$, we can write

$$\sum_{n\leqslant x} \prod_{1\leqslant i\leqslant k} r(n+a_i,Q) = \bigg(\frac{\zeta(\ell/2)\Gamma(\ell/2)}{(2\pi)^{\ell/2}}\bigg)^{k} S + O\big(x^{(\ell/2-1)(k-1)+1+\ell/4-\delta_\ell+\varepsilon}\big), \tag{6.1}$$

where

$$S := \sum_{n\leqslant x} \prod_{1\leqslant i\leqslant k} (n+a_i)^{\ell/2-1}\sigma(n+a_i, Q).$$

Inserting the series expansion for $\sigma(n,Q)$ and using simple relation

$$(n+a_1)^{\ell/2-1}\cdots(n+a_k)^{\ell/2-1} = n^{(\ell/2-1)k} + O_{a_1,\dots,a_k}\big(n^{(\ell/2-1)k-1}\big),$$

it follows that

$$\begin{aligned} S = \sum_{q_1=1}^{\infty}\cdots\sum_{q_k=1}^{\infty}\sum_{h_1=1}^{q_1}{}^{*}\cdots\sum_{h_k=1}^{q_k}{}^{*} &\frac{S(h_1Q/q_1)\cdots S(h_kQ/q_k)}{(q_1\cdots q_k)^{\ell}}\mathrm{e}\bigg(-\frac{h_1a_1}{q_1}-\cdots-\frac{h_ka_k}{q_k}\bigg) \\ \times \sum_{n\leqslant x} n^{(\ell/2-1)k}\mathrm{e}\bigg\{-n\bigg(\frac{h_1}{q_1}+\cdots+\frac{h_k}{q_k}\bigg)\bigg\} &+ O\big(x^{(\ell/2-1)k}\big). \end{aligned}$$

By virtue of (1.12), the infinite series

$$\sum_{q_1=1}^{\infty}\cdots\sum_{q_k=1}^{\infty}\sum_{h_1=1}^{q_1}{}^{*}\cdots\sum_{h_k=1}^{q_k}{}^{*} \frac{S(h_1Q/q_1)\cdots S(h_kQ/q_k)}{(q_1\cdots q_k)^{\ell}}\mathrm{e}\bigg(-\frac{h_1a_1}{q_1}-\cdots-\frac{h_ka_k}{q_k}\bigg)$$

is absolutely convergent. Since

$$\sum_{n\leqslant x} n^{(\ell/2-1)k} = \frac{x^{(\ell/2-1)k+1}}{(\ell/2-1)k+1} + O\big(x^{(\ell/2-1)k}\big),$$

the contribution of $(q_1,\dots,q_k,h_1,\dots,h_k)$ with $h_1/q_1+\cdots+h_k/q_k\in\mathbb{Z}$ to S is

$$C_Q(a_1,\dots,a_k)x^{(\ell/2-1)k+1} + O\big(x^{(\ell/2-1)k}\big). \tag{6.2}$$

By using (2.2), partial summation and the fact $\|h_1/q_1+\cdots+h_k/q_k\|\geqslant(q_1\cdots q_k)^{-1}$, the contribution of $(q_1,\dots,q_k,h_1,\dots,h_k)$ with $h_1/q_1+\cdots+h_k/q_k\notin\mathbb{Z}$ to S is

$$\ll x^{(\ell/2-1)k}\sum_{q_1=1}^{\infty}\cdots\sum_{q_k=1}^{\infty}\frac{\min\{x, q_1\cdots q_k\}}{(q_1\cdots q_k)^{\ell/2-1}} \ll x^{(\ell/2-1)k+\eta_\ell(\varepsilon)}, \tag{6.3}$$

where we have used the following estimate

$$\min\{x, q_1\cdots q_k\} \leqslant \begin{cases} x^{1/2+\varepsilon}(q_1\cdots q_k)^{1/2-\varepsilon} & \text{if } \ \ell=5, \\ x^{\varepsilon}(q_1\cdots q_k)^{1-\varepsilon} & \text{if } \ \ell=6,7, \\ q_1\cdots q_k & \text{if } \ \ell\geqslant 8. \end{cases}$$

Now Theorem 3 follows from (6.2) and (6.3), by noticing that

$$(\ell/2-1)(k-1)+1+\ell/4-\delta_\ell+\varepsilon \leqslant (\ell/2-1)k+\eta_\ell(\varepsilon) \quad (\ell\geqslant 5).$$

7. Proof of Corollary 1

By (1.5) and (1.6), we have, for $n \leqslant x$,

$$\prod_{i=1}^{k} \sigma_{\ell/2-1}(n+a_i) = \bigg(\frac{\zeta(\ell/2)\Gamma(\ell/2)}{(2\pi)^{\ell/2}}\bigg)^k \prod_{i=1}^{k} r(n+a_i, Q) + O\bigg(x^{(k-d)(\ell/2-1)/2} \sum_{d=1}^{k-1} \sum_{\{i_1,\dots,i_d\}\subset\{1,\dots,k\}} \prod_{j=1}^{d} r(n+a_{i_j}, Q)\bigg).$$

Now Theorem 3 implies the required result since

$$(k-d)(\ell/2-1)/2 + (\ell/2-1)d + 1 \leqslant (\ell/2-1)(k-1/2) + 1 \leqslant (\ell/2-1)k.$$

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